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Low-Mach-number asymptotics of the Navier-Stokes equations

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Abstract. A multiple-time scale, single-space scale asymptotic analysis of the compressible Navier-Stokes equations reveals how the heat-release rate and heat conduction affect the zeroth-order global thermodynamic pressure, the divergence of velocity and the material change of density at low-Mach-numbers. The asymptotic analysis identifies the acoustic time change of the heat-release rate as the dominant source of sound in low-Mach-number flow. The viscous and buoyancy forces enter the computation of the second-order 'incompressible' pressure in low-Mach-number flow in a similar way as they enter the pressure computation in incompressible flow, except for a nonzero velocity-divergence constraint. If the flow equations are averaged over an acoustic wave period, the averaged velocity tensor describes the net acoustic effect on the averaged flow field. Removing acoustics from the equations altogether leads to the low-Mach-number equations, which allow for large temperature and density changes as opposed to the Boussinesq equations.

Keywords: Navier-Stokes equations, low-Mach-number limit, multiple scales, asymptotics.

1. Introduction

Speaking, singing and playing a music instrument are pleasant examples of compressible low-Mach-number flow, unless they are considered as noise. Acoustics in gases and liquids is not only a daily experience, but also has great scientific and technological significance. Aerodynamic-noise regulations have become more restrictive due to public demands caused by increased transport of persons and goods and by increased environmental sensitivity. Thus, aero-acoustics has become a key issue in the design of airplanes, helicopters, trains, cars, engines, gas-turbines, etc. The available computational power and emerging numerical methods have recently led to computational aero-acoustics (CAA) as a new branch of computational fluid dynamics (CFD). Like CFD, CAA offers great potential to complement theoretical and experimental aero-acoustic research. Therefore, Sir James Lighthill foresees a 'second golden age of aero-acoustics' [1].

Flow can generate sound, but also the opposite mechanism is possible. In acoustic streaming [2], sound generates flow: in a fluid adjacent to a wall, an acoustic standing wave induces a steady secondary vortex flow due to the presence of viscosity. This astonishing effect can be used to enhance heat and mass transfer [3].

The interaction of heat-release and sound is decisive for the operation of rocket motors, gas-turbines, etc. According to Rayleigh's criterion, pressure is amplified, if heat-release and pressure fluctuate in phase. This acoustic instability has to be controlled to avoid destruction of motors, turbines, etc. due to pressure amplification [4], [5], [6].

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Since in many practical applications the sound generation is primarily determined by the slow vorticity fluctuations in a turbulent flow or by entropy fluctuations in a slow turbulent flame, there has been considerable interest in low-Mach-number aero-acoustics [7] and thermo-acoustics [6].

Asymptotic analysis has been performed to get better mathematical insight into the governing equations and better physical understanding of the mechanisms they describe, as the Mach-number M goes to zero. Low-Mach-number asymptotics is used by Klainerman and Majda [8] for the Euler equations and by Kreiss *et al*. [9] for the Navier-Stokes equations to prove the convergence of the compressible-flow solutions to the incompressible-flow solutions for $M \to 0$ under certain conditions. Majda [10] employs low-Mach-number asymptotics to derive the equations for low-Mach-number combustion. The resonant interaction of small-amplitude periodic high-frequency acoustic waves with entropy and vorticity waves is studied by Hunter *et al.* [11], Majda *et al*. [12], Almgren [13] and other authors using the method of multiple scales. Rehm and Baum [14] derive the low-Mach-number limit of the compressible Euler equations for buoyant inviscid flow with heat release. For the low-Mach-number limit of the compressible Navier-Stokes equations, Fedorchenko [15] provides a number of exact solutions. Employing multiple-time and space scale expansions, Zank and Matthaeus [16] derive low-Mach-number equations from the compressible Navier-Stokes equations. The single-time scale, multiple-space scale asymptotic analysis by Klein [17] yields insight into the low-Mach-number limit behaviour of the compressible Euler equations and has been used to develop a numerical method for low-Mach-number flow with long-wave acoustics. For small turbulent Mach-numbers, a single-space scale, multiple-time scale asymptotics allows Ristorcelli [18] to distinguish advective and acoustic modes in a vortical source of sound and to find a closure for the compressible aspects of the source terms in Lighthill's acoustic analogy.

The objective of the present work is to give insight into the compressible Navier-Stokes equations at low-Mach-number, when slow flow is affected by acoustic effects in a bounded region over a long time. We may think of a modern gas-turbine combustor, where acoustic waves are reflected at the turbine inlet and the upstream wall and interact many times with the turbulent flame [6]. Another application in mind is a closed piston-cylinder system, in which the isentropic compression due to a slow piston motion is modified by acoustic waves. These are generated by the piston start and propagate back and forth many times, because they are reflected at the cylinder end and piston. In that problem, we have one length scale, say the initial distance between piston and cylinder end, and two time scales: the long time it takes the slow flow, *i.e.* the piston, to travel one length scale and the short time it takes for an acoustic wave to travel one length scale. Opposed to a regular low-Mach-number expansion, an asymptotic analysis with two time scales and one space scale, together with the analytical method of characteristics, avoids secular terms and allows G. H. Schneider [19] (*cf*. W. Schneider [20], pp. 235–240) and Klein and Peters [21] to account for the cumulative acoustic effects in the inert and reacting piston-cylinder problems, respectively. Using a similar asymptotic approach, Rhadwan and Kassoy [22] investigate the acoustic response due to boundary heating in a confined inert gas. Here, the multiple-time scale and single-space scale asymptotic analysis is employed to get a better understanding of the low-Mach-number limit of the compressible Navier-Stokes equations and to derive simplified equations, which account for the net effect of periodic acoustic waves on slow flow over a long time. The present work is based on the author's habilitation thesis [23].

The Euler and Navier-Stokes equations for compressible flow are stated in Section 2. The multiple-time scale, single-space scale asymptotic analysis in Section 3 gives insight into the low-Mach-number limit behaviour of the compressible-flow equations. In Section 4, an equation accounting for the net acoustic effect and the low-Mach-number equations are derived, and the latter are related to the Boussinesq and incompressible-flow equations. In Section 5, conclusions are stated.

2. Euler and Navier-Stokes equations

The conservation laws of mass, momentum and energy for an inviscid and viscous flow are called the Euler and Navier-Stokes equations, respectively, in computational fluid dynamics.

The Navier-Stokes equations in differential conservative form read as follows:

continuity equation:

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{u}^*) = 0, \tag{1}$$

momentum equation:

$$\frac{\partial \rho^* \mathbf{u}^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{u}^* \mathbf{u}^*) + \nabla^* p^* = \mathbf{G}^*, \tag{2}$$

energy equation:

$$\frac{\partial \rho^* E^*}{\partial t^*} + \nabla^* \cdot (\rho^* H^* \mathbf{u}^*) = Q^*. \tag{3}$$

All dimensional quantities are denoted by superscript *, *e.g.* time t^*. ρ^*, p^*, $\mathbf{u}^*$ denote the density, pressure and velocity vector, respectively. The vector operators $\nabla^*\cdot$ and ∇^* are the divergence and gradient, respectively. In Cartesian coordinates, the components of the position vector, velocity vector and nabla operator are given by $\mathbf{x}^* = (x^*, y^*, z^*)^T$, $\mathbf{u}^* = (u^*, v^*, w^*)^T$ and $\nabla^* = (\partial/\partial x^*, \partial/\partial y^*, \partial/\partial z^*)^T$, respectively.

$\mathbf{G}^*$ represents the sum of external forces. Here we consider viscous and buoyancy forces

$$\mathbf{G}^* = \nabla^* \cdot \boldsymbol{\tau}^* + \rho^* \mathbf{g}^*. \tag{4}$$

For a Newtonian fluid, the shear stress tensor $\boldsymbol{\tau}^*$ is given by

$$\boldsymbol{\tau}^* = \mu^*(\nabla^* \mathbf{u}^* + (\nabla^* \mathbf{u}^*)^T) - \tfrac{2}{3}\mu^* \nabla^* \cdot \mathbf{u}^* \, \mathbf{I} \tag{5}$$

with the (dynamic) viscosity μ^*, which depends on temperature T^*. $\mathbf{I}$ is the unit tensor. The gravitational acceleration vector $\mathbf{g}^*$ is directed opposite to the radial unit vector $\mathbf{e}_r$ in spherical coordinates with the gravity constant $g^* = 9.81\,\mathrm{m/s^2}$ on the earth's surface:

$$\mathbf{g}^* = -g^* \mathbf{e}_r. \tag{6}$$

Further

$$E^* = e^* + \tfrac{1}{2}|\mathbf{u}^*|^2 \tag{7}$$

denotes the total energy per unit mass, *i.e.* the sum of internal energy e^* and kinetic energy $\frac{1}{2}|\mathbf{u}^*|^2$;

$$H^* = E^* + \frac{p^*}{\rho^*} \tag{8}$$

is the total enthalpy.

Q^* is the sum of the work done by the external forces and of the heat released by external sources per unit volume and per unit time, *i.e.*

$$Q^* = \nabla^* \cdot (\boldsymbol{\tau}^* \cdot \mathbf{u}^*) + \rho^* \mathbf{g}^* \cdot \mathbf{u}^* + \nabla^* \cdot (\kappa^* \nabla^* T^*) + \rho^* q^*. \tag{9}$$

The Fourier law is assumed for the heat-conduction term, in which the heat-conduction coefficient κ^* depends on temperature T^*. The heat-release rate q^* might be due to chemical reactions.

If viscosity and heat conduction are negligible, *i.e.* if $\mu^* \equiv 0$ and $\kappa^* \equiv 0$, the Euler equations are obtained. Thereby, the type of the equations is changed from hyperbolic-parabolic to hyperbolic [24]. If buoyancy is negligible, *i.e.* if $g^* \equiv 0$, $\mathbf{G}^*$ and Q^* are simplified.

For a perfect gas, the thermodynamic quantities are related by the equations of state

$$p^* = \rho^* R^* T^*, \tag{10}$$

$$e^* = c_v^* T^* \tag{11}$$

with the specific gas constant $R^* = c_p^* - c_v^*$ and the specific heats at constant pressure and volume c_p^* and c_v^*, respectively. The ratio of specific heats is

$$\gamma = \frac{c_p^*}{c_v^*}. \tag{12}$$

For air at standard conditions, $\gamma = 1{\cdot}4$.

3. Low-Mach-number asymptotics

In the following subsections 1 and 2, a suitable nondimensionalization and a multiple-time scale, single-space scale asymptotic analysis, respectively, are provided for the compressible Navier-Stokes equations at low-Mach-numbers.

3.1. Nondimensionalization

We nondimensionalize the Equations (1–11) by using reference quantities denoted by the subscript ∞, *e.g.* farfield or stagnation conditions, and a typical length scale L^* of the considered flow. The thermodynamic reference quantities are assumed to be related by the equation of state (10) for a perfect gas. We define the nondimensional quantities by:

$$\begin{gathered}\rho = \frac{\rho^*}{\rho^*_\infty}, \quad p = \frac{p^*}{p^*_\infty}, \quad \mathbf{u} = \frac{\mathbf{u}^*}{u^*_\infty}, \quad T = \frac{T^*}{T^*_\infty}, \quad \mu = \frac{\mu^*}{\mu^*_\infty}, \quad \kappa = \frac{\kappa^*}{\kappa^*_\infty}, \\ \mathbf{x} = \frac{\mathbf{x}^*}{L^*}, \quad t = \frac{t^*}{L^*/u^*_\infty}, \quad e = \frac{e^*}{p^*_\infty/\rho^*_\infty}, \quad E = \frac{E^*}{p^*_\infty/\rho^*_\infty}, \quad H = \frac{H^*}{p^*_\infty/\rho^*_\infty}.\end{gathered} \tag{13}$$

The reference quantities are chosen such that the nondimensional flow quantities remain of order O(1) for any low-reference-Mach-number

$$M_\infty = \frac{u_\infty^*}{\sqrt{\gamma p_\infty^* / \rho_\infty^*}}. \tag{14}$$

To avoid the dependence on γ, we shall work with:

$$\tilde{M} = \frac{u_\infty^*}{\sqrt{p_\infty^* / \rho_\infty^*}} = \sqrt{\gamma} M_\infty. \tag{15}$$

Note that with other nondimensionalizations

$$\frac{u^*}{\sqrt{p_\infty^* / \rho_\infty^*}} = \frac{u^*}{u_\infty^*} \tilde{M} \to 0 \quad \text{for} \quad \tilde{M} \to 0,$$

and

$$\frac{p^*}{\rho_\infty^* u_\infty^{*2}} = \frac{p^*}{p_\infty^*} \frac{1}{\tilde{M}^2} \to \infty \quad \text{for} \quad \tilde{M} \to 0.$$

Using the relations (13) in the equations of Section 2, we may write the nondimensional Navier-Stokes equations as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{16}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \frac{1}{\tilde{M}^2} \nabla p = \mathbf{G}, \tag{17}$$

where $\mathbf{G} = 1/\mathrm{Re}_\infty \nabla \cdot \boldsymbol{\tau} + (1/\mathrm{Fr}_\infty^2)\rho(-\mathbf{e}_r)$ $\mathrm{Re}_\infty = \rho_\infty^* u_\infty^* L^* / \mu_\infty^*$ is the Reynolds number and $\mathrm{Fr}_\infty = u_\infty^* / \sqrt{g^* L^*}$ is the Froude number. Also,

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho H \mathbf{u}) = Q, \tag{18}$$

where

$$Q = \frac{\tilde{M}^2}{\mathrm{Re}_\infty} \nabla \cdot (\boldsymbol{\tau} \cdot \mathbf{u}) + \frac{\tilde{M}^2}{\mathrm{Fr}_\infty^2} \rho(-\mathbf{e}_r) \cdot \mathbf{u} + \frac{\gamma}{(\gamma - 1)\,\mathrm{Re}_\infty \mathrm{Pr}_\infty} \nabla \cdot (\kappa \nabla T) + \rho q.$$

$$\mathrm{Pr}_\infty = \frac{c_p^* \mu_\infty^*}{\kappa_\infty^*}$$

is the Prandtl number and $q = q^* / (u_\infty^* p_\infty^* / L^* \rho_\infty^*)$ is the nondimensional heat-release rate.

The nondimensional expressions of the total energy per unit mass and the total enthalpy are

$$E = e + \tilde{M}^2 \tfrac{1}{2} |\mathbf{u}|^2, \tag{19}$$

$$H = E + \frac{p}{\rho}. \tag{20}$$

The nondimensional equations of state for a perfect gas read

$$p = \rho T, \tag{21}$$

$$e = \frac{1}{\gamma - 1} T. \tag{22}$$

Using Equations (19), (21) and (22), we may express the pressure in terms of the conservative variables ρ, $\rho\mathbf{u}$ and ρE by

$$p = (\gamma - 1)\left[\rho E - \tilde{M}^2 \frac{1}{2} \frac{|\rho\mathbf{u}|^2}{\rho}\right]. \tag{23}$$

Assuming $\mathrm{Pr} = c_p^* \mu^*/\kappa^* = \mathrm{Pr}_\infty$ and c_p^* = const, we obtain $\kappa^*/\kappa_\infty^* = \mu^*/\mu_\infty^*$. Then

$$\kappa = \mu. \tag{24}$$

The viscosity might for example be determined by the Sutherland law

$$\mu = T^{\frac{3}{2}} \frac{1 + S}{T + S} \tag{25}$$

with $S = T_{\mathrm{ref}}^*/T_\infty^*$ for air at standard conditions, where $T_{\mathrm{ref}}^* = 110\,\mathrm{K}$.

3.2. Asymptotic analysis

As mentioned in the introduction, we are interested in slow flow affected by acoustic effects in a confined gas over a long time. Therefore, we introduce the fast acoustic time scale

$$\tau = \frac{t^*}{L^*/\sqrt{\frac{p_\infty^*}{\rho_\infty^*}}} = \frac{t}{\tilde{M}}. \tag{26}$$

The flow and acoustic time scales are illustrated in Figures 1 and 2 for the two characteristics $\mathrm{d}x^*/\mathrm{d}t^* = u_\infty^*$ and $\mathrm{d}x^*/\mathrm{d}t^* = \sqrt{p_\infty^*/\rho_\infty^*} = c_\infty^*/\sqrt{\gamma}$, respectively. Whereas the flow time scale is determined by the time it takes the reference flow to travel one length scale, the acoustic time scale corresponds to the time it takes to travel one length scale at the reference speed of sound divided by $\sqrt{\gamma}$.

In the two-time scale, single-space scale low-Mach-number asymptotic analysis, each flow variable is expanded as *e.g.*, the pressure

$$p(\mathbf{x}, t, \tilde{M}) = p_0(\mathbf{x}, t, \tau) + \tilde{M} p_1(\mathbf{x}, t, \tau) + \tilde{M}^2 p_2(\mathbf{x}, t, \tau) + O(\tilde{M}^3) \tag{27}$$

with

$$\tau = \frac{t}{\tilde{M}} \quad \text{and} \quad \tilde{M} = \frac{u_\infty^*}{\sqrt{p_\infty^*/\rho_\infty^*}}.$$

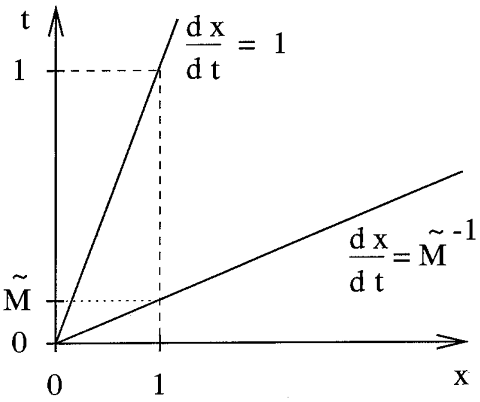

Figure 1. Characteristics in flow time.

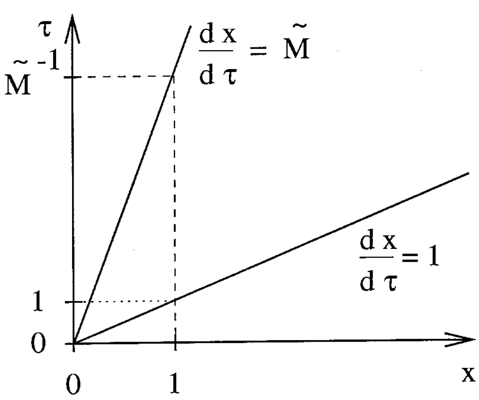

Figure 2. Characteristics in acoustic time.

The time derivative at constant $\mathbf{x}$ and $\tilde{M}$ involves the flow time derivative $\partial/\partial t$ and the acoustic time derivative $\partial/\partial\tau$:

$$\left.\frac{\partial p}{\partial t}\right|_{\mathbf{x},\tilde{M}} = \left(\frac{\partial}{\partial t} + \frac{1}{\tilde{M}}\frac{\partial}{\partial\tau}\right)[p_0 + \tilde{M}p_1 + \tilde{M}^2 p_2 + O(\tilde{M}^3)]. \tag{28}$$

The leading, first- and second-order continuity equations read

$$\frac{\partial\rho_0}{\partial\tau} = 0, \tag{29}$$

$$\frac{\partial\rho_1}{\partial\tau} + \frac{\partial\rho_0}{\partial t} + \nabla\cdot(\rho\mathbf{u})_0 = 0, \tag{30}$$

$$\frac{\partial\rho_2}{\partial\tau} + \frac{\partial\rho_1}{\partial t} + \nabla\cdot(\rho\mathbf{u})_1 = 0. \tag{31}$$

Equation (29) implies that ρ_0 does not depend on the acoustic time scale, *i.e.* $\rho_0 = \rho_0(\mathbf{x}, t)$. Expanding the density ρ, momentum density $(\rho\mathbf{u})$ and velocity $\mathbf{u}$ similar to the pressure in (27) and using the identity $(\rho\mathbf{u}) = \rho\mathbf{u}$, we obtain the relations $(\rho\mathbf{u})_0 = \rho_0\mathbf{u}_0$, $(\rho\mathbf{u})_1 = \rho_1\mathbf{u}_0 + \rho_0\mathbf{u}_1$, and $(\rho\mathbf{u})_2 = \rho_2\mathbf{u}_0 + \rho_1\mathbf{u}_1 + \rho_0\mathbf{u}_2$.

The leading, first- and second-order momentum equations are derived as

$$\nabla p_0 = 0, \tag{32}$$

$$\frac{\partial(\rho\mathbf{u})_0}{\partial\tau} + \nabla p_1 = 0, \tag{33}$$

$$\frac{\partial(\rho\mathbf{u})_1}{\partial\tau} + \frac{\partial(\rho\mathbf{u})_0}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u})_0 + \nabla p_2 = \mathbf{G}_0 \tag{34}$$

with $\mathbf{G}_0 = 1/\mathrm{Re}_\infty\nabla\cdot\boldsymbol{\tau}_0 + (1/\mathrm{Fr}_\infty^2)\rho_0(-\mathbf{e}_r)$. $p_0(t,\tau)$ does not depend on $\mathbf{x}$ because of Equation (32).

The leading, first- and second-order energy equations yield

$$\frac{\partial(\rho E)_0}{\partial \tau} = 0, \tag{35}$$

$$\frac{\partial(\rho E)_1}{\partial \tau} + \frac{\partial(\rho E)_0}{\partial t} + \boldsymbol{\nabla} \cdot (\rho H \mathbf{u})_0 = Q_0, \tag{36}$$

$$\frac{\partial(\rho E)_2}{\partial \tau} + \frac{\partial(\rho E)_1}{\partial t} + \boldsymbol{\nabla} \cdot (\rho H \mathbf{u})_1 = Q_1 \tag{37}$$

with

$$Q_0 = \frac{\gamma}{(\gamma - 1)\,\mathrm{Re}_\infty\,\mathrm{Pr}_\infty} \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T)_0 + (\rho q)_0$$

and

$$Q_1 = \frac{\gamma}{(\gamma - 1)\,\mathrm{Re}_\infty\,\mathrm{Pr}_\infty} \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T)_1 + (\rho q)_1.$$

Equation (35) implies that $(\rho E)_0 = (\rho E)_0(\mathbf{x}, t)$. Since the work done by the viscous and buoyancy forces is of order $O(\tilde{M}^2)$, the zeroth- and first-order energy-source terms Q_0 and Q_1 are governed by heat-conduction and heat-release rate only, provided the Prandtl number Pr_∞ is of order $O(1)$ and the Froude number squared Fr_∞^2 is of order $O(\mathrm{Re}_\infty\,\mathrm{Pr}_\infty)$, provided that the ratio $(\gamma - 1)/\gamma$ is of order $O(1)$ in both cases. However, if the Reynolds number Re_∞ is of the order $O(\tilde{M}^2)$, *i.e.* the reference pressure p_∞^* is of the order of the viscous force per unit area $O(\mu_\infty^* u_\infty^*/L^*)$, or if the Froude number Fr_∞ is of the order $O(\tilde{M})$, *i.e.* the reference pressure p_∞^* is of the order of the hydrostatic pressure $O(\rho_\infty^* g^* L^*)$, then the work done by the viscous or buoyancy forces, respectively, will also contribute to the zeroth-order energy source term Q_0.

For the equations of state, the asymptotic expansion yields

$$p_0 = (\gamma - 1)(\rho E)_0, \tag{38}$$

$$p_1 = (\gamma - 1)(\rho E)_1, \tag{39}$$

$$p_2 = (\gamma - 1)[(\rho E)_2 - \tfrac{1}{2}\rho_0 u_0^2], \tag{40}$$

$$T_0 = \frac{p_0}{\rho_0}, \tag{41}$$

$$T_1 = \frac{p_1 - \rho_1 T_0}{\rho_0}, \tag{42}$$

$$T_2 = \frac{p_2 - \rho_1 T_1 - \rho_2 T_0}{\rho_0}. \tag{43}$$

Using the consequences of Equations (32) and (35) in (38), we obtain for the zeroth-order pressure

$$p_0(t, \tau) = (\gamma - 1)(\rho E)_0(\mathbf{x}, t). \tag{44}$$

Consequently, the zeroth-order pressure p_0 and the zeroth-order total energy density $(\rho E)_0$ only depend on the flow time t, and we get

$$p_0(t) = (\gamma - 1)(\rho E)_0(t). \tag{45}$$

Using the consequence of Equation (29), we may simplify the first-order momentum Equation (33) to

$$\frac{\partial \mathbf{u}_0}{\partial \tau} + \frac{1}{\rho_0}\nabla p_1 = 0. \tag{46}$$

With $\rho_0 H_0 = (\rho E)_0 + p_0 = (\gamma/(\gamma - 1))p_0$, (38), and (39), the first-order energy Equation (36) becomes

$$\frac{\partial p_1}{\partial \tau} + \gamma p_0 \nabla \cdot \mathbf{u}_0 = (\gamma - 1)Q_0 - \frac{\mathrm{d}p_0}{\mathrm{d}t}. \tag{47}$$

Subtracting the divergence of (46) multiplied by γp_0 from the acoustic time derivative of (47), *i.e.* $\partial/\partial\tau$ (47) $-\gamma p_0 \nabla\cdot$ (46), we have (since $\partial/\partial\tau$ and $\nabla\cdot$ commute)

$$\frac{\partial^2 p_1}{\partial \tau^2} - \nabla \cdot (c_0^2\, \nabla p_1) = (\gamma - 1)\frac{\partial(\rho q)_0}{\partial \tau}. \tag{48}$$

Note that $c_0^2 = \gamma p_0(t)/\rho_0(\mathbf{x}, t)$ depends on $\mathbf{x}$ and t, but not on τ. Thus, Equation (48) represents an inhomogeneous linear wave equation with nonconstant coefficient c_0^2. Its source is due to the acoustic time change of the zeroth-order heat-release rate. The source of (48) is not affected by heat conduction, because the acoustic time derivative of the zeroth-order temperature $T_0 = p_0(t)/\rho_0(\mathbf{x}, t)$ vanishes. The work done by the viscous and buoyancy forces does not affect the first-order pressure p_1, unless the conditions $\mathrm{Re}_\infty = O(\tilde{M}^2)$ or $\mathrm{Fr}_\infty = O(\tilde{M})$, respectively, hold.

If the zeroth-order speed of sound c_0 is approximated by the ambient speed of sound c_∞ (with the ambient chosen as reference state), Equation (48) constitutes the basic equation of thermo-acoustics to describe acoustic effects in combustion [5], [25]. The acoustic time change of the heat-release rate constitutes the dominant thermo-acoustic source in combustion, as the right-hand side of (48) is governed by the monopole source $(\gamma - 1)\partial(\rho q)_0/\partial\tau$ [5], [25]. The governing equation of thermo-acoustics, *i.e.* the simplification of (48) just mentioned, can also be derived by simplifying Lighthill's acoustic analogy for combustion [25]. That equation can be solved analytically by means of Green's function [26].

The zeroth-order pressure p_0 can be determined by integration of Equation (47) over the volume V of the computational domain [14]:

$$\frac{\mathrm{d}p_0}{\mathrm{d}t}\int_V \mathrm{d}V + \gamma p_0 \int_{\partial V} \mathbf{u}_0 \cdot \mathbf{n}\, \mathrm{d}A = (\gamma - 1)\int_V Q_0\, \mathrm{d}V - \int_V \frac{\partial p_1}{\partial \tau}\, \mathrm{d}V. \tag{49}$$

Knowing $\mathbf{u}_0$ at the boundary of the computational domain, knowing the heat conduction and heat-release rate $\int_V Q_0\, \mathrm{d}V$ in the computational domain and assuming that the integrated effect of the acoustic time change of the first-order pressure is negligible, *i.e.* $\int_V (\partial p_1/\partial\tau)\, \mathrm{d}V = 0$, we may solve the ODE (49) for $p_0(t)$, starting from an initial condition $p_0(0)$.

Equation (47), derived from the first-order energy equation, has an interesting consequence for the divergence of the zeroth-order velocity:

$$\nabla \cdot \mathbf{u}_0 = \frac{\gamma - 1}{\gamma p_0} Q_0 - \frac{1}{\gamma p_0} \left[\frac{\mathrm{d}p_0}{\mathrm{d}t} + \frac{\partial p_1}{\partial \tau} \right]. \tag{50}$$

Thus, the divergence of $\mathbf{u}_0$ is affected by the heat conduction and heat-release rate Q_0, the time change of the global pressure p_0 and the acoustic time change of the first-order pressure p_1.

Using the product rule in the first-order continuity Equation (30) and inserting (50), we find that the zeroth-order density change along the zeroth-order path line reads

$$\frac{\partial \rho_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \rho_0 = -\frac{\gamma - 1}{c_0^2} Q_0 + \frac{1}{c_0^2} \frac{\mathrm{d}p_0}{\mathrm{d}t} + \frac{\partial}{\partial \tau} \left(\frac{p_1}{c_0^2} - \rho_1 \right). \tag{51}$$

Thus, the zeroth-order density of a fluid particle is changed by the heat conduction and heat-release rate Q_0, the global pressure time change $\mathrm{d}p_0/\mathrm{d}t$ and by the acoustic time change of the first-order enthropy, *i.e.* $\partial/\partial\tau(p_1/c_0^2 - \rho_1)$.

4. Low-Mach-number equations

If the first-order continuity and energy equations and the second-order momentum equation are averaged over an acoustic wave period, the averaged velocity tensor describes the net acoustic effect on the averaged flow field. Removal of the acoustics altogether leads to the low-Mach-number equations. Their derivation and their relation to the Boussinesq and incompressible flow equations are indicated here.

We assume acoustic waves with period T_a such that $(\rho_1, (\rho\mathbf{u})_1, (\rho E)_1)^T(\mathbf{x}, t, \tau) = (\rho_1, (\rho\mathbf{u})_1, (\rho E)_1)^T(\mathbf{x}, t, \tau + T_a)$. Integrating Equation (51) over a period T_a of the acoustic wave, using $\int_0^{T_a} \partial/\partial\tau(p_1/c_0^2 - \rho_1)\,\mathrm{d}\tau = 0$, we obtain the simplification

$$\frac{\partial \rho_0}{\partial t} + \bar{\mathbf{u}}_0 \cdot \nabla \rho_0 = -\frac{\gamma - 1}{c_0^2} \bar{Q}_0 + \frac{1}{c_0^2} \frac{\mathrm{d}p_0}{\mathrm{d}t}, \tag{52}$$

where the overbar denotes acoustic time averaging, *i.e.* $\bar{\mathbf{u}}_0(\mathbf{x}, t) = T_a^{-1} \int_0^{T_a} \mathbf{u}_0(\mathbf{x}, t, \tau)\,\mathrm{d}\tau$ for the averaged velocity.

Similarly, Equation (50) can be simplified by integration:

$$\nabla \cdot \bar{\mathbf{u}}_0 = \frac{\gamma - 1}{\gamma p_0} \bar{Q}_0 - \frac{1}{\gamma p_0} \frac{\mathrm{d}p_0}{\mathrm{d}t}. \tag{53}$$

Integrating the second-order momentum Equation (34) over the acoustic wave period T_a, we get

$$\frac{\partial \rho_0 \bar{\mathbf{u}}_0}{\partial t} + \nabla \cdot (\rho_0 \overline{\mathbf{u_0 u_0}}) + \nabla \bar{p}_2 = \bar{\mathbf{G}}_0, \tag{54}$$

where the averaged velocity tensor $\overline{\mathbf{u_0 u_0}} = T_a^{-1} \int_0^{T_a} \mathbf{u}_0(\mathbf{x}, t, \tau)\mathbf{u}_0(\mathbf{x}, t, \tau)\,\mathrm{d}\tau$ describes the net acoustic effect on the averaged flow field. The solution of the averaged momentum equation

(54) requires the knowlege of $\mathbf{u}_0(\mathbf{x}, t, \tau)$ over an acoustic wave period to determine $\overline{\mathbf{u_0 u_0}}$. In a numerical solution, we might be able to choose the numerical time step Δt equal to the acoustic wave period T_a and solve the inhomogeneous wave equation (48) by subdividing T_a into a number of acoustic time steps $\Delta\tau$ to obtain $p_1(\mathbf{x}, t, \tau + m\Delta\tau)$, $m = 1, \ldots, T_a/\Delta\tau$. Then, the numerical solution of (46) yields $\mathbf{u}_0(\mathbf{x}, t, \tau + m\Delta\tau)$, $m = 1, \ldots, T_a/\Delta\tau$. Finally, the averaged velocity tensor $\overline{\mathbf{u_0 u_0}}$ can be obtained by numerical integration.

If the averaged velocity tensor $\overline{\mathbf{u_0 u_0}}$ is approximated by the tensor $\bar{\mathbf{u}}_0\bar{\mathbf{u}}_0$ of the averaged velocities, even the net acoustic effect is removed from (54) and we obtain the momentum equation in the zero-Mach-number limit:

$$\frac{\partial \rho_0 \bar{\mathbf{u}}_0}{\partial t} + \nabla \cdot (\rho_0 \bar{\mathbf{u}}_0 \bar{\mathbf{u}}_0) + \nabla \bar{p}_2 = \bar{\mathbf{G}}_0. \tag{55}$$

Assuming ρ_0 and the right-hand side of Equation (53) to be known, we may determine the averaged velocity $\bar{\mathbf{u}}_0$ and the averaged pressure $\bar{p}_2$ analogously to incompressible flow. For any smooth $\bar{\mathbf{u}}_0$, the divergence of Equation (55) divided by ρ_0 yields a Poisson-type equation to determine $\bar{p}_2$. At this point, the viscous and buoyancy forces described by $\bar{\mathbf{G}}_0$ come into play; $\bar{\mathbf{u}}_0$ is determined by the divergence constraint (53).

The averaged first-order continuity equation (30) yields

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \bar{\mathbf{u}}_0) = 0. \tag{56}$$

Using (56) to express $\nabla \cdot \bar{\mathbf{u}}_0$ and using the equation of state $p_0(t) = \rho_0(\mathbf{x}, t)T_0(\mathbf{x}, t)$, we see that energy Equation (53) becomes.

$$\frac{\gamma}{\gamma - 1}\rho_0 \left[\frac{\partial T_0}{\partial t} + \bar{\mathbf{u}}_0 \cdot \nabla T_0\right] - \frac{\mathrm{d}p_0}{\mathrm{d}t} = \bar{Q}_0. \tag{57}$$

If the viscous forces are neglected in the momentum source $\bar{\mathbf{G}}_0$, Equations (56), (55), (57) are Rehm and Baum's low-Mach-number equations [14] obtained by an asymptotic analysis of the compressible Euler equations for slow heat addition to describe flows in fires. The pressure $p_0(t)$ is determined by Equation (49) integrated over the acoustic wave period [14]. Similar equations are used in low-Mach-number combustion [10], [27], [28], where the pressure across a flame is almost constant. The low-Mach-number equations are valid for arbitrary temperature and density changes governed by the equation of state $p_0(t) = \rho_0(\mathbf{x}, t)T_0(\mathbf{x}, t)$. The equations admit internal waves like gravity waves [14].

However, acoustic waves are removed from the low-Mach-number Equations (55)–(57), because the equations were derived by integration of (30), (34) and (36) over the acoustic wave period. The removal of acoustics from the low-Mach-number equations can be realized directly by single-time and space-scale asymptotic analysis, *i.e.* by the expansion of each flow variable as, *e.g.*, the pressure

$$p(\mathbf{x}, t, \tilde{M}) = p_0(\mathbf{x}, t) + \tilde{M} p_1(\mathbf{x}, t) + \tilde{M}^2 p_2(\mathbf{x}, t) + O(\tilde{M}^3), \tag{58}$$

without considerations on an acoustic time or length scale [10]. Since the zeroth- and first-order quantities are governed by the same equations, the first-order quantities need not to be considered separately, *e.g.* $\tilde{M} p_1$ in (58) can be omitted. Then, the first-order continuity and energy equations corresponding to (30) and (36) and the second-order momentum equation corresponding to (34) yield the low-Mach-number Equations (55–57). Thus, either the acoustic time scale (*cf.* Section 3) or the acoustic length scale [17] has additionally to be taken into account in the asymptotic analysis to describe acoustic effects in low-Mach-number flow.

Contrary to the low-Mach-number equations, the intermediate Equations (54), (56) and (57) take into account the acoustic effect of the nonlinearity on the averaged flow in the momentum equation. It will be interesting to investigate the significance of (54) instead of (55), because, like the low-Mach-number equations, the intermediate Equations (54), (56) and (57) can be solved more easily than the compressible Navier-Stokes equations.

Since the zeroth-order pressure $p_0(t)$ serves as the mean pressure in the energy equation, it represents the global thermodynamic pressure part. As the second-order pressure $p_2(\mathbf{x}, t)$ is determined by the momentum equation as the pressure complying with the constraint (53) on the velocity-divergence analogously to incompressible flow, $\tilde{M}^2 p_2(\mathbf{x}, t)$ may be called the 'incompressible' pressure part. The low-Mach-number momentum equation (55) is coupled to the low-Mach-number energy equation (57) via the density ρ_0, viscosity $\mu(T_0)$ and equation of state $p_0 = \rho_0 T_0$, whereas the first-order momentum and energy equations with acoustics included (46) and (47) are directly coupled by the first-order pressure $p_1(\mathbf{x}, t, \tau)$. Since $p_1(\mathbf{x}, t, \tau)$ is governed by the inhomogeneous wave equation (48), $\tilde{M} p_1(\mathbf{x}, t, \tau)$ can be identified as the acoustic pressure part. These roles of the pressure are also identified by a single-time scale, multiple-space-scale low-Mach-number asymptotics for the Euler equations [17].

If only small temperature and density changes are allowed and if p_0 is assumed to be constant, the low-Mach-number Equations (55–57) simplify to the Boussinesq equations [14]. In the Boussinesq equations, the momentum equation is coupled to the energy equation by the buoyancy force depending on the reduced temperature $T_0 - T_\infty$.

We obtain the incompressible Euler and Navier-Stokes equations by neglecting the buoyancy force and thereby the coupling between the momentum and energy equations present in the Boussinesq equations. The velocity and pressure can then be determined independently of the temperature [14]. We may see the elliptic character of p_2 as for the low-Mach-number and Boussinesq equations by applying the divergence operator to the momentum equation and using the relevant constraint on $\nabla \cdot \bar{\mathbf{u}}_0$. The divergence of the velocity is zero for the incompressible and the Boussinesq equations according to the continuity equation, whereas $\nabla \cdot \bar{\mathbf{u}}_0$ depends on the heat conduction and heat-release rate Q_0 and the global pressure time change $\mathrm{d}p_0/\mathrm{d}t$ according to the constraint (53) derived from the energy equation for the low-Mach-number equations.

5. Conclusions

A low-Mach-number asymptotic analysis of the compressible Navier-Stokes equations yields insight into the roles of the zeroth-, first- and second-order pressures, into the dependence on heat-release rate, heat conduction, viscous and buoyancy forces and into the relation to the low-Mach-number equations. The net effect of periodic acoustic waves on the averaged slow flow can be accounted for by the averaged velocity tensorin the momentum equation.

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